The σ-algebra generated by a collection of subsets.

Question: Let X be a set and let $\mathcal{B}$ be a subset of $\mathcal{P}(X)$. How to describe the σ-algebra generated by $\mathcal{B}$?

Answer:

"Implicit" Construction

According to definition, the σ-algebra generated by $\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{B}$. As $\mathcal{P}(X)$ is a σ-algebra containing B, we can take $\mathcal{S}$ to be the intersection of all the σ-algebras (which do exist) containing $\mathcal{B}$. Then we can check (why?) that $\mathcal{S}$ is a σ-algebra containing $\mathcal{B}$. It remains to show that among σ-algebras containing $\mathcal{B}$, $\mathcal{S}$ is the smallest one. In fact, let $\mathcal{D}$ be any σ-algebra containing B, from the definition of $\mathcal{S}$, we know that $\mathcal{S} \subset \mathcal{D}$.

"Explicit" Construction?

The approach above does construct the σ-algebra generated by $\mathcal{B}$ implicitly. If you want it explicitly, it turns out to be a non-trivial question.

In case $\mathcal{B}$ contains only finitely many elements (with each element being a subset of X), it is not hard to prove that the σ-algebra generated by $\mathcal{B}$ also contains finitely many elements (why?). Besides, if $\mathcal{B} = \{B_i \colon 1 \leq i \leq N\}$, then each element in the σ-algebra generated by $\mathcal{B}$ is of the form

$$\bigcup_{j=1}^{2^N} \bigcap_{i=1}^{2^N} A_{i,j},$$

where each $A_{i,j}$ satisfies $A_{i,j} \in \mathcal{B}$ or $A_{i,j}^c \in \mathcal{B}$.

If $\mathcal{B}$ contains infinitely many elements, one naive guess is that each element in the σ-algebra generated by $\mathcal{B}$ is of the form

$$\bigcup_{j=1}^{\infty} \bigcap_{i=1}^{\infty} A_{i,j},$$

where each $A_{i,j}$ satisfies $A_{i,j} \in \mathcal{B}$ or $A_{i,j}^c \in \mathcal{B}$.

It is surely true that each of the above $\bigcup_{j=1}^{\infty} \bigcap_{i=1}^{\infty} A_{i,j}$ is in the σ-algebra generated by $\mathcal{B}$. In general, however, it might not be true that each element in the σ-algebra generated by $\mathcal{B}$ is of the form $\bigcup_{j=1}^{\infty}\bigcap_{i=1}^{\infty}A_{i,j}$.

One idea to give some constructive description of the σ-algebra generated by $\mathcal{B}$ is like this, although I do not think it makes this σ-algebra much easier to understand.

Start with $\mathcal{B}$ be a subset of $\mathcal{P}(X)$. Without loss of generality, we assume that for any $A \in \mathcal{B}$, we have $A^c \in \mathcal{B}$.

Let $E_0 = \mathcal{B}$.

Let E_1 be made up of elements which is at most countably union of elements in E_0

Let E_2 be made up of elements which is either an element in E_1 or the complement of an element in E_1.

Let E_3 be made up of elements which is at most countably union of elements in E_2.

Let E_4 be made up of elements which is either an element in E_3 or the complement of an element in E_3.

$\cdots$

Let E_ω be the union of all those E_i s with $i \in \mathbb{N}_{\geq 1}$. As for elements in E_ω, we can easily show that their complements is still in E_ω. However, for countably many elements in E_ω, there is no way we can ensure that the union of countably many elements in E_ω is still in E_ω. Thus we need to keep going.

Let $E_{\omega+1}$ be made up of elements which is at most countably union of elements in E_ω.

Let $E_{\omega+2}$ be made up of elements which is either an element in $E_{\omega+1}$ or the complement of an element in $E_{\omega+1}$.

$\cdots$

Let $E_{2\omega}$ be the union of all those previous occured $E_{\text{something}}$ s.

Then we can keep going and the lower index will be $2\omega+1$, $2\omega+2$, $\cdots$, 3ω, $\cdots$, 4ω, $\cdots$, ω^2, $\cdots$, ω^3, $\cdots$, ω^4, $\cdots$, $\cdots$, ω^ω, $\cdots$, ω^{ω^ω}, $\cdots$, ω_1. Here w_1 is the first ordinality that is uncountable. Note that we already assumed the Axiom of Choice in this paragraph.

Roughly speaking, we are using "ordinal number" to give certain description of elements in the σ-algebra generated by $\mathcal{B}$.

Assuming the Axiom of Choice, we can apply transfinite induction on the ordinal number w_1. Note that a countable union of countable sets is still countable, by taking E to be the union of all those E_α where the ordinality α satisfies $\alpha < \omega_1$, one can see (need quite some work on set theory, ordinality, etc) that this E describes the σ-algebra generated by $\mathcal{B}$.

Remark: I do not think this "description" makes the σ-algebra generated by $\mathcal{B}$ easier to understand though.